

Tricyclic graphs with exactly two main eigenvalues *

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Abstract

An eigenvalue of a graph G is called a main eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero. In this paper, all connected tricyclic graphs with exactly two main eigenvalues are determined.

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1 Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $A(G)$ the adjacency matrix of G . The eigenvalues of G are those of $A(G)$. An eigenvalue of a graph G is called a main eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero. It is well known that a graph is regular if and only if it has exactly one main eigenvalue.

A long-standing problem posed by Cvetkovic ([2]) is that of how to characterize graphs with exactly $k(k \geq 2)$ main eigenvalues. Hagos [3] gave an alternative characterization of graphs with exactly two main eigenvalues. Recently, Hou and Zhou [4] characterized the trees with exactly two main eigenvalues.

A vertex of a graph G is said to be pendant if it has degree one. Denote by C_n and P_n the cycle and path of order n , respectively. A connected graph is said to be tricyclic (resp., unicyclic and bicyclic), if $|E(G)| = |V(G)| + 2$ (resp., $|E(G)| = |V(G)|$ and $|E(G)| = |V(G)| + 1$). Hou and Tian [5] showed that the graphs C_r^k for some positive integers k, r with $r \geq 3$, where C_r^k is the graph obtained from C_r by attaching $k > 0$ pendant vertices to every vertex of C_r , are the only connected unicyclic graphs with exactly two main eigenvalues. Hu et al. [6] and Shi [7] characterized all connected bicyclic graphs with exactly two main eigenvalues independently. This paper will continue the line of this research and determine all connected tricyclic graphs with exactly two main eigenvalues.

For any tricyclic graph G , the base of G , denoted by G_B is the minimal tricyclic subgraph of G . Clearly, G_B is the unique tricyclic subgraph of G containing no pendent vertex, and G can be obtained from G_B by attaching trees to some vertices of G_B . It follows from [8] that there are 8 types of bases for tricyclic graphs, say, $\mathcal{T}_i, i = 1, \dots, 8$, which are depicted in Fig. 1.

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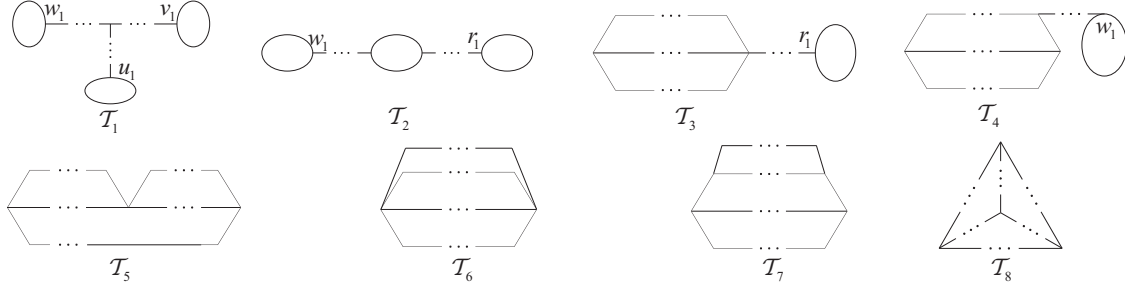


Figure 1: The 8 types of bases for tricyclic graphs.

2 Preliminaries

In this section, we will present some notations and known results which will be used in the next section. The reader is referred to [1] for any undefined notation and terminology on graphs in this paper.

Let G be a graph. As usual, we denote by $d(v) = d_G(v)$ and $N(v) = N_G(v)$ the degree of vertex v and the set of all neighbors of v in G . Let

$$S(v) = \sum_{u \in N(v)} d(u). \quad (2.1)$$

A graph G is called 2-walk (a, b) -linear if there exist unique rational numbers a, b such that

$$S(v) = ad(v) + b \quad (2.2)$$

holds for every vertex $v \in V(G)$.

An internal path of G is a walk $v_0 v_1 \dots v_s$ such that the vertices v_0, v_1, \dots, v_s are distinct, $d(v_0) > 2, d(v_s) > 2$, and $d(v_i) = 2$ for $0 < i < s$. An internal path is called an internal cycle if $v_0 = v_s$. If R is a path or a cycle of G , the length of R , denoted by $l(R)$, is defined as the number of edges of R .

Lemma 2.1 ([3]). *A graph G has exactly two main eigenvalues if and only if G is 2-walk (a, b) -linear.*

Lemma 2.2 ([5]). *Let G be a 2-walk (a, b) -linear graph. Then both a and b are integers.*

Lemma 2.3 ([6]). *Let G be a 2-walk (a, b) -linear graph and v, u be two vertices of G with unequal degree $d(v), d(u)$, respectively. Then*

$$a = \frac{S(v) - S(u)}{d(v) - d(u)}, \quad b = \frac{d(u)S(v) - d(v)S(u)}{d(v) - d(u)}. \quad (2.3)$$

In the following, for convenience, we always assume that \mathcal{G} is the set of tricyclic graphs with exactly two main eigenvalues, x is a pendent vertex of G (if exist). For each $G \in \mathcal{G}$, let G_0 be the graph obtained from G by deleting all pendent vertices. From the proof Lemmas 3.1-3.7 in [6], we know that those Lemmas are also hold for tricyclic graphs. Hence we have the following two Lemmas.

Lemma 2.4 *Let $G \in \mathcal{G}$ and $R = x_1 x_2 \dots x_t$ be an internal path of length at least 2 in G . Then $l(R) \leq 3$. In particular, if $l(R) = 3$, then there exists no path $Q = y_1 y_2 y_3$ in G such that $d(y_1) = d(y_3) = d(x_1)$ and $d(y_2) = 2$.*

Lemma 2.5 *Let $G \in \mathcal{G}$ and $v \in V(G_0)$. Then*

- (i) $G_0 \in \mathcal{T}_i, i = 1, \dots, 8$ (see Fig. 1);
- (ii) $d(v) = d_{G_0}(v)$ or $a + b$;
- (iii) if G has at least one pendent vertex, then $S(x) = a + b \geq 3$ and $a \geq 2$;
- (iv) for a cycle $C = x_1x_2 \dots x_tx_1$ of G with $d_{G_0}(x_1) \geq 3$, $d_{G_0}(x_t) = 2$, if G has at least one pendent vertex, then there is an integer $i \in \{1, 2, \dots, t\}$ such that $d(x_i) \neq a + b$.

3 Tricyclic graphs with exactly two main eigenvalues

In this section, we will determine all tricyclic graphs with exactly two main eigenvalues. By Lemma 2.1, it is sufficient to determine all 2-walk (a, b) -linear tricyclic graphs.

Lemma 3.1 *Let $G \in \mathcal{G}$ has at least one pendent vertex and let $R = x_1x_2 \dots x_t$ be an internal path or an internal cycle of length at least 3 in G_0 with $d_{G_0}(x_1) = d_{G_0}(x_t) \in \{3, 4, 6\}$ or $d_{G_0}(x_1) = 3, d_{G_0}(x_t) = 5$. Then*

- (i) $d(x_2) = d(x_3) = \dots = d(x_{t-1}) \in \{2, a + b\}$ and $d(x_1) = d(x_t)$;
- (ii) if $d(x_2) = 2$, then $l(R) = 3$;
- (iii) if R is a cycle with $d_{G_0}(x_1) \in \{3, 4, 6\}$, then $l(R) = 3$ and $d(x_2) = d(x_3) = 2$. In particular, if $d_{G_0}(x_1) = 3$, then $a = 2$;
- (iv) if R is a cycle with $d_{G_0}(x_1) = 5$, then $l(R) = 3$, $d(x_2) = d(x_3) \in \{2, 3\}$. In particular, if $d(x_2) = d(x_3) = 3$, then $d(x_1) = 5$ and $a = 3, b = 0$.

Proof. (i) By way of contradiction, assume that there is an integer $i \in \{2, 3, \dots, t-2\}$ such that $d(x_i) \neq d(x_{i+1})$. Without loss of generality, suppose that i is the smallest integer such that $d(x_i) \neq d(x_{i+1})$. By Lemma 2.5 (ii), we may assume that $d(x_i) = 2$ and $d(x_{i+1}) = a + b$. Hence $d(x_2) = d(x_3) = \dots = d(x_i) = 2$. Applying (2.3) with $(v, u) = (x_{i+1}, x)$, we have

$$a = \frac{S(x_{i+1}) - S(x)}{d(x_{i+1}) - d(x)} = \frac{a + b - 2 + 2 + d(x_{i+2}) - (a + b)}{a + b - 1} = \frac{d(x_{i+2})}{a + b - 1}.$$

This together with Lemma 2.5 (iii) implies that

$$d(x_{i+2}) = a(a + b - 1) \geq 2(a + b - 1) \geq a + b + 1 > \max\{a + b, 3\}. \quad (3.1)$$

By Lemma 2.5 (ii), we have $d(x_j) \in \{a + b, 2\}$ for $2 \leq j \leq t-1$. Thus $x_{i+2} \in \{x_1, x_t\}$.

If $d_{G_0}(x_1) = d_{G_0}(x_t) = 3$, then $d(x_{i+2}) \in \{d(x_1), d(x_t)\} \subseteq \{3, a + b\}$, contrary to (3.1).

If $d_{G_0}(x_1) = d_{G_0}(x_t) = 4$, then $d(x_1), d(x_t) \in \{4, a + b\}$. It follows from (3.1) that $d(x_{i+2}) = a(a + b - 1) = 4$. This together with Lemma 2.5 (iii) implies that $a = 2, b = 1$. Note that $d(x_2) = 2$. Then $S(x_2) = 5$ by (2.2). On the other hand, $d_{G_0}(x_1) = 4 > a + b$, so $d(x_1) = 4$ by Lemma 2.5 (ii). Thus by (2.1), $S(x_2) = d(x_1) + d(x_3) = 4 + d(x_3) > 5$, a contradiction.

If $d_{G_0}(x_1) = d_{G_0}(x_t) = 6$, with a similar argument of the case $d_{G_0}(x_1) = d_{G_0}(x_t) = 4$, we will get a contradiction again.

If $d_{G_0}(x_1) = 3, d_{G_0}(x_t) = 5$, then by (3.1) $d(x_{i+2}) = d(x_t) = d_{G_0}(x_t) = 5$ and $a(a + b - 1) = 5$. It contradicts the fact that $a \geq 2, a + b \geq 3$.

Hence $d(x_2) = d(x_3) = \dots = d(x_{t-1}) \in \{2, a + b\}$. Therefore $S(x_2) = S(x_{t-1})$ by (2.2). On the other hand, by (2.1), $S(x_2) = d(x_2) - 2 + d(x_1) + d(x_3), S(x_{t-1}) = d(x_{t-1}) - 2 + d(x_t) + d(x_{t-3})$. It follows that $d(x_1) = d(x_t)$.

(ii) Suppose contrary that $l(R) \geq 4$. Then $d(x_3) = d(x_4) = 2$ by (i). Applying (2.3) with $(v, u) = (x_3, x)$, we have $a = 4 - (a + b)$, which is impossible by Lemma 2.5 (iii).

(iii) By Lemma 2.5 (ii), we have $d(x_2) \in \{2, a+b\}$. If $d(x_2) = a+b$, then $d(x_i) = a+b$ for $2 \leq i \leq t-1$ by (i). Thus $d(x_1) = d_{G_0}(x_1) \neq a+b$ by Lemma 2.5 (iv). Applying (2.3) with $(v, u) = (x_2, x)$, we have

$$a = \frac{a+b-2+d(x_1)+a+b-(a+b)}{a+b-1} = 1 + \frac{d(x_1)-1}{a+b-1} = 1 + \frac{d_{G_0}(x_1)-1}{a+b-1}. \quad (3.2)$$

Note that $d_{G_0}(x_1) = 3, 4$ or 6 and $d_{G_0}(x_1) \neq a+b \geq 3$. It follows from (3.2) that a can not be an integer. It contradicts Lemma 2.2. Hence $d(x_2) = 2$. Therefore $l(R) = 3$ and $d(x_3) = d(x_2) = 2$.

In particular, if $d_{G_0}(x_1) = 3$, then $a = 2 + d(x_1) - (a+b)$ by applying (2.3) with $(v, u) = (x_2, x)$. If $d(x_1) = a+b$, then $a = 2$. If $d(x_1) = d_{G_0}(x_1) = 3$, then $a = 5 - (a+b)$. It follows from Lemmas 2.2 and 2.5 (iii) that $a = 2$.

(iv) By Lemma 2.5 (ii), $d(x_2) \in \{2, a+b\}$. If $d(x_2) = 2$, then $l(R) = 3$ and $d(x_3) = 2$ by (i) and (ii). If $d(x_2) = a+b$, then $d(x_i) = a+b$ for $2 \leq i \leq t-1$ by (i). So $d(x_1) = d_{G_0}(x_1) = 5 \neq a+b$ by Lemma 2.5 (iv). Applying (2.3) with $(v, u) = (x_2, x)$, we have $a = 1 + \frac{4}{a+b-1}$. Thus $a+b = 3$ and $a = 3$ by Lemmas 2.2 and 2.5 (iii). Suppose that $l(R) \geq 4$. Then $d(x_2) = d(x_3) = d(x_4) = a+b = 3$ by (ii). Applying (2.3) with $(v, u) = (x_3, x)$, we have $a = 2$. It is a contradiction. Therefore $l(R) = 3$. \square

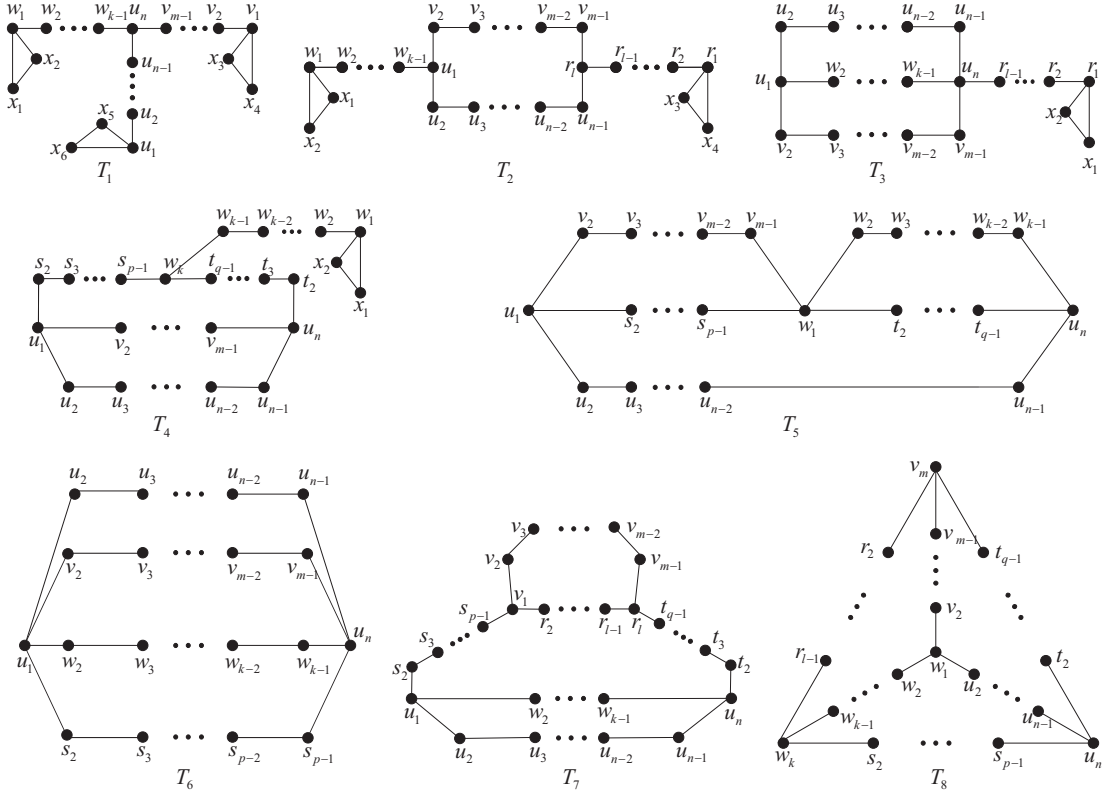


Figure 2: The graphs T_i for $i = 1, \dots, 8$.

Lemma 3.2 Let $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_1$ (see Fig. 1). Then $G = H_i$ for $i = 1, 2$ (see Fig. 3).

Proof. If $G_0 \in \mathcal{T}_1$, then $d_{G_0}(u_1), d_{G_0}(v_1), d_{G_0}(w_1) \in \{3, 4, 5, 6\}$ and so each cycle of G_0 has the length of 3 by Lemmas 2.4 and 3.1. Hence $G_0 = T_1$ (see Fig. 2), where $n, m, k \geq 1$. For convenience, we set $w_k = v_m = u_n$.

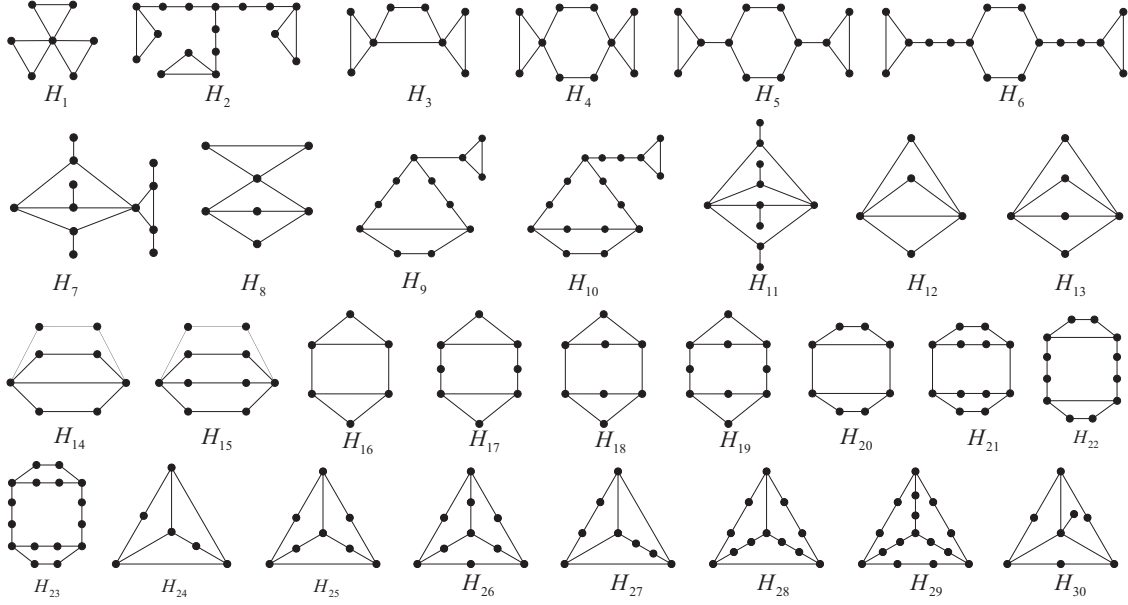


Figure 3: The graphs H_i for $i=1, \dots, 30$.

Case 1. $n = m = k = 1$. If G has no pendent vertex, then $G = H_1$ (see Fig. 3). By (2.3), H_1 is 2-walk $(1, 6)$ -linear. If G has at least one pendent vertex, say x , then $x \in N(u_1)$ since $d(x_i) = 2$ for $i = 1, \dots, 6$ by Lemma 3.1 (iii). It follows from Lemma 2.5 (ii) that $d(u_1) = a + b > 6$. Applying (2.3) with $(v, u) = (u_1, x)$, we have $a = \frac{a+b-6+12-(a+b)}{a+b-1} < 2$. This contradicts Lemma 2.5 (iii).

Case 2. $n \geq 2$. We consider the following two cases:

Subcase 1. G has no pendent vertex. Then $S(x_1) = 2 + d(w_1) = S(x_5) = 5$ by (2.1) and (2.2). Hence $d(w_1) = 3$. It implies that $k \geq 2$. Similarly, we have $d(v_1) = 3$ and $m \geq 2$. By Lemma 2.4, we have $n, m, k \in \{2, 4\}$. Without loss of generality, suppose that $n \geq m \geq k$. If $n = 2$, then $m = k = 2$. Hence $S(u_1) = 7, S(u_2) = 9$ by (2.1). On the other hand, $d(u_1) = d(u_2) = 3$, so $S(u_1) = S(u_2)$ by (2.2), a contradiction. Hence $n = 4$. Similarly, we have $m = k = 4$. Therefore $G = H_2$ (see Fig. 3). By (2.3), H_2 is 2-walk $(1, 3)$ -linear.

Subcase 2. G has at least one pendent vertex. In this case, we show that there is no such graph with exactly two main eigenvalues. Since $d_{G_0}(u_1) = 3$, we have $a = 2$ by Lemma 3.1 (iii). So $d(x_i) = 2$ for $i = 1, \dots, 6$ by Lemma 3.1 (iii) and (iv).

We claim that $m, k \geq 2$. Otherwise, let $k = 1$. Then $S(x_1) = 2 + d(w_1) = S(x_5) = 2 + d(u_1)$ by (2.1) and (2.2). It follows from Lemma 2.5 (ii) and the fact that $d_{G_0}(u_1) \neq d_{G_0}(w_1)$ that $d(w_1) = d(u_1) = a + b$. Hence $S(u_1) = S(w_1)$ by (2.2). By (2.1),

$$S(u_1) = a + b - 3 + 4 + d(u_2), S(w_1) = \begin{cases} a + b - 5 + 8 + d(u_{n-1}), & \text{if } m = 1, \\ a + b - 4 + 4 + d(v_{m-1}) + d(u_{n-1}), & \text{if } m \geq 2. \end{cases}$$

If $m = 1$, then $d(u_2) = d(u_{n-1}) + 2$. This is impossible by Lemma 3.1 (i). If $m \geq 2$, then $d(u_2) + 1 = d(v_{m-1}) + d(u_{n-1})$. Obviously, $u_2 = u_{n-1}$ when $n = 3$ and $d(u_2) = d(u_{n-1})$ when $n \geq 4$ by Lemma 3.1 (i). Hence $d(v_{m-1}) = 1$, it contradicts the fact that $d(v_{m-1}) \geq d_{G_0}(v_{m-1}) = 2$. Therefore $k \geq 2$. Dually, we have $m \geq 2$.

Note that $d(x_1) = d(x_3) = d(x_5)$, we have $S(x_1) = 2 + d(w_1) = S(x_3) = 2 + d(v_1) = S(x_5) = 2 + d(u_1)$ by (2.1) and (2.2). It implies that $d(w_1) = d(v_1) = d(u_1) = 3$ or $a + b$.

If $d(u_1) = 3$. Applying (2.3) with $(v, u) = (x_5, x)$, we have $a = 5 - (a + b)$. So $a = 2, b = 1$. Furthermore, $S(u_1) = 4 + d(u_2) = 7$ by (2.1) and (2.2). Thus $d(u_2) = 3$. It follows from

Lemma 3.1 (i) that $d(u_i) = 3$ for $2 \leq i \leq n-1$. Similarly, we have $d(v_i) = d(w_j) = 3$ for $2 \leq i \leq m-1$ and $2 \leq j \leq k-1$. Note that $d_{G_0}(u_n) = a+b = 3$. We have $d(u_n) = 3$ and $S(u_n) = 7$ by (2.2). On the other hand, $S(u_n) = 9$ by (2.1), a contradiction.

If $d(u_1) = a+b \neq 3$, then $a+b \geq 4$ by Lemma 2.5 (iii). Applying (2.3) with $(v, u) = (u_1, x)$, we have $2 = \frac{a+b-3+4+d(u_2)-(a+b)}{a+b-1}$. Note that $d_{G_0}(u_2) = 2$ or 3 , we have $d(u_2) = 2(a+b) - 3 > \max\{a+b, d_{G_0}(u_2)\}$. It contradicts Lemma 2.5 (ii). \square

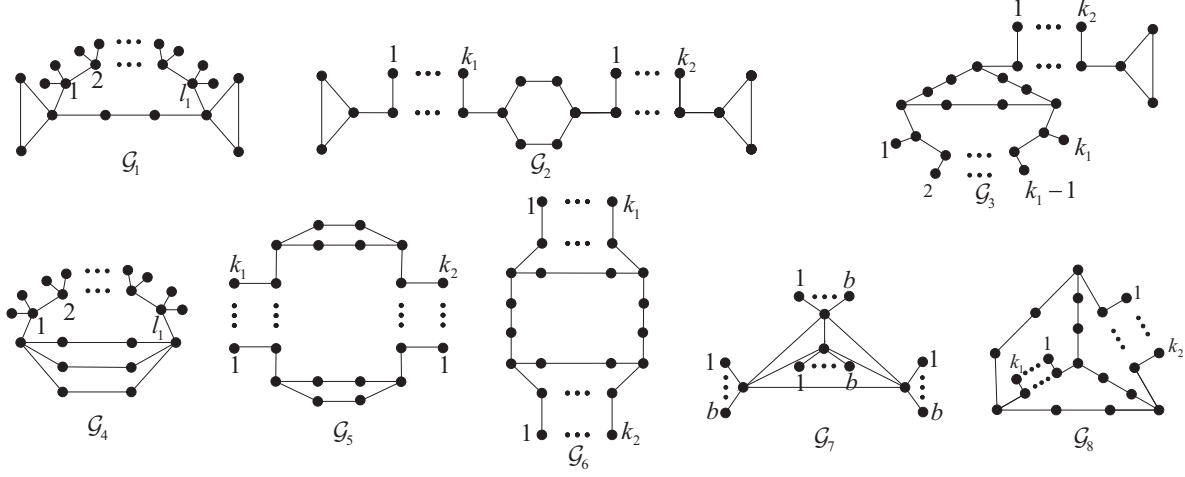


Figure 4: The graphs \mathcal{G}_i for $i=1, \dots, 8$, where $l_1, b \geq 1$ and $\max\{k_1, k_2\} \geq 1$

Lemma 3.3 Let $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_2$ (see Fig. 1). Then $G = H_i$ for $i = 3, 4, 5, 6$ (see Fig. 3) or $G \in \mathcal{G}_j$ for $j = 1, 2$ (see Fig. 4).

Proof. If $G_0 \in \mathcal{T}_2$, then $d_{G_0}(w_1), d_{G_0}(r_1) \in \{3, 4\}$ and so each cycle of G_0 has the length of 3 by Lemmas 2.4 and 3.1 (iii). Hence $G_0 = T_2$ and $d(x_i) = 2$ for $i = 1, \dots, 4$ (see Fig. 2), where $k, l \geq 1, n, m \geq 2$. For convenience, we set $w_k = v_1 = u_1$ and $v_m = u_n = r_l$. By (2.1) and (2.2), $S(x_1) = 2 + d(w_1) = S(x_3) = 2 + d(r_1)$. Hence $d(w_1) = d(r_1)$.

If G has no pendent vertex, then $k, l \in \{1, 2, 4\}$ and $m, n \in \{2, 4\}$ by Lemma 2.4.

First, let $k = 1$. Then $d(w_1) = d(r_1) = 4$. So $l = 1$. Therefore $G = H_i$ for $i = 3, 4$ (see Fig. 3). By (2.3), H_3 and H_4 are 2-walk (2, 2)-linear and 2-walk (1, 4)-linear, respectively.

Next, let $k = 2$. Then $d(w_1) = d(r_1) = 3$. By (2.1) and (2.2), $S(r_1) = 4 + d(r_2) = S(w_1) = 7$. So $d(r_2) = 3$ and $l = 2$. Similarly, $S(u_1) = 3 + d(u_2) + d(v_2) = S(w_1) = 7$. So $d(u_2) = d(v_2) = 2$. Note that $n, m = 2$ or 4 , we have $n = m = 4$ and $d(u_3) = d(v_3) = 2$. Therefore, $G = H_5$ (see Fig. 3). By (2.3), H_5 is 2-walk (2, 1)-linear.

Finally, let $k = 4$. With a similar argument of the case $k = 2$, we have $G = H_6$ is 2-walk (1, 3)-linear (see Fig. 3).

If G has at least one pendent vertex. We consider the following two cases:

Case 1. $k = l = 1$. Then $d(w_1) = d(r_1) = 4$ or $a+b$ by Lemma 2.5 (ii).

If $d(w_1) = 4$. Applying (2.3) with $(v, u) = (x_1, x)$, we have $a = 6 - (a+b)$. It follows from Lemmas 2.2 and 2.5 (iii) that $a+b = 3$ or 4 .

If $a+b = 3$, then $a = 3, b = 0$. So $S(w_1) = 4 + d(u_2) + d(v_2) = 12$ by (2.1) and (2.2). By Lemma 2.5 (ii), $d(u_2), d(v_2) \in \{2, 4, a+b\}$. Thus $d(u_2) = d(v_2) = 4$. It implies that $n = m = 2$, which is impossible since G is simple.

If $a+b = 4$, then $a = b = 2$. So $S(w_1) = 4 + d(u_2) + d(v_2) = 10$ by (2.1) and (2.2). By Lemma 2.5 (ii), $d(u_2), d(v_2) \in \{2, 4\}$. Without loss of generality, suppose that

$d(u_2) = 2, d(v_2) = 4$. Then $d(v_i) = 4$ for $2 \leq i \leq m-1$ by Lemma 3.1 (i). For the vertex u_2 , $S(u_2) = 4 + d(u_3) = 6$ by (2.1) and (2.2). So $d(u_3) = 2$. Thus $n \geq 4$. Hence $n = 4$ by Lemma 3.1 (ii). Therefore $G \in \mathcal{G}_1$ (see Fig. 4), where $l_1 \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_1$ is 2-walk (2, 2)-linear.

If $d(w_1) = a + b > d_{G_0}(u_1) = 4$. In this case, we show that there is no such graph with exactly two main eigenvalues. Applying (2.3) with $(v, u) = (x_1, x)$ and $(v, u) = (w_1, x)$, respectively. We have $a = 2$ and $a = \frac{a+b-4+4+d(u_2)+d(v_2)-(a+b)}{a+b-1}$, respectively. Hence $d(u_2) + d(v_2) = 2(a + b) - 2$. By Lemma 2.5 (ii) and the fact that $d(u_n) = d(r_1) = d(w_1) = a + b$, we have $d(u_2), d(v_2) \in \{2, a + b\}$.

If $d(u_2) = 2, d(v_2) \in \{2, a + b\}$ or $d(u_2) = a + b, d(v_2) = 2$, then $a + b = 3$ or 4. It contradicts the fact that $a + b > 4$.

If $d(u_2) = d(v_2) = a + b$, then $2(a + b) - 2 = 2(a + b)$, a contradiction.

Case 2. $k \geq 2$. Then $a = 2$ by Lemma 3.1 (iii). By Lemmas 2.5 (ii), $d(w_1) = d(r_1) = 3$ or $a + b$.

We first show that $d(w_1) = d(r_1) = 3$. Otherwise, let $d(w_1) = d(r_1) = a + b \neq 3$. Then $a + b \geq 4$ by Lemma 2.5 (iii). Applying (2.3) with $(v, u) = (w_1, x)$, we have $2 = \frac{a+b-3+4+d(w_2)-(a+b)}{a+b-1}$. Note that $a+b \geq 4$ and $d_{G_0}(w_2) = 2$ or 3. We have $d(w_2) = 2(a+b)-3 > \max\{a+b, d_{G_0}(w_2)\}$. It contradicts Lemma 2.5 (ii). Hence $d(w_1) = d(r_1) = 3$. It implies that $l \geq 2$.

Next, applying (2.3) with $(v, u) = (x_1, x)$, we have $a = 5 - (a + b)$. So $a + b = 3$ and $a = 2, b = 1$. For the vertex w_1 , $S(w_1) = 4 + d(w_2) = 7$ by (2.1) and (2.2). So $d(w_2) = 3$. Thus $d(w_i) = 3$ for $2 \leq i \leq k-1$ by Lemma 3.1 (i). Note that $d_{G_0}(w_k) = 3$. We have $d(w_k) = 3$ by Lemma 2.5 (ii). Hence $d(w_i) = 3$ for $1 \leq i \leq k$.

Similarly, we have $d(r_j) = 3$ for $1 \leq j \leq l$.

For the vertex w_k , $S(w_k) = 3 + d(u_2) + d(v_2) = 7$. Note that $d(u_2), d(v_2) \in \{2, 3\}$ by Lemma 2.5 (ii). We have $d(u_2) = d(v_2) = 2$. It follows from Lemmas 2.4 and 3.1 that $m = n = 4$ and $d(u_3) = d(v_3) = 2$.

Therefore $G \in \mathcal{G}_2$ (see Fig. 4), where $\max\{k_1, k_2\} \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_2$ is 2-walk (2, 1)-linear. \square

Lemma 3.4 *Let $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_3$ (see Fig. 1). Then $G = H_7$ (see Fig. 3).*

Proof. If $G_0 \in \mathcal{T}_3$, then $d_{G_0}(r_1) \in \{3, 5\}$ and so the cycle of G_0 has the length of 3 by Lemmas 2.4 and 3.1. Hence $G_0 = T_3$ (see Fig. 2), where $n, m, k \geq 2, l \geq 1$. For convenience, we set $u_1 = v_1 = w_1$ and $u_n = v_m = w_k = r_l$.

We first show that G contains at least one pendent vertex. On the contrary, suppose that G has no pendent vertex. Then $n, m, k, l \leq 4$ by Lemma 2.4. Without loss of generality, suppose that $n \geq m \geq k$. If $l = 1$, then $S(x_1) = 7$ and $S(u_2) = 5$ or 8 by (2.1). So $S(u_2) \neq S(x_1)$. On the other hand, $S(u_2) = S(x_1)$ by (2.2), a contradiction. If $l \geq 2$, then $d(u_n) = 4, d(r_1) = 3$. So $S(x_1) = 5$ and $S(u_{n-1}) = 6$ or 7 by (2.1). On the other hand, $S(x_1) = S(u_{n-1})$ by (2.2), a contradiction. Therefore, G has at least one pendent vertex. We consider the following two cases:

Case 1. $l = 1$. By Lemma 3.1 (iv), $d(x_1) = d(x_2) = 3$ or $d(x_1) = d(x_2) = 2$.

If $d(x_1) = d(x_2) = 3$, then $a = 3, b = 0, d(u_n) = 5$ by Lemma 3.1 (iv). So $d(u_1) = 3 \neq d(u_n)$ by Lemma 2.5 (ii). It follows from Lemma 3.1 (i) that $n, m, k \leq 3$. Without loss of generality, suppose that $n = m = 3, k = 2$ or 3. Then $S(u_3) = 6 + d(u_2) + d(v_2) + d(w_{k-1}) = 15$ by (2.1) and (2.2). Note that $d(u_2), d(v_2), d(w_{k-1}) = 2$ or 3 by Lemma 2.5 (ii). We have $d(u_2) = d(v_2) = d(w_{k-1}) = 3$. So $S(u_1) = 6 + d(u_2) = 9$ by (2.1) and (2.2). Thus $d(w_2) = 3$. It implies that $k = 3$. Therefore $G = H_7$ (see Fig. 3). By (2.3), H_7 is 2-walk (3, 0)-linear.

If $d(x_1) = d(x_2) = 2$. We show that in this case there is no such graph with exactly two main eigenvalues. We consider the following two cases:

Subcase 1. $\max\{n, m, k\} \geq 4$. Without loss of generality, suppose that $n \geq 4$. Then $d(u_1) = d(u_n)$ and $d(u_i) = d(u_2)$ for $2 \leq i \leq n-1$ by Lemma 3.1 (i). Note that $d_{G_0}(u_1) \neq d_{G_0}(u_n)$. We have $d(u_1) = d(u_n) = a + b \geq d_{G_0}(u_n) = 5$ by Lemma 2.5 (ii). Hence

$$S(u_1) = a + b - 3 + d(u_2) + d(v_2) + d(w_2), \quad S(u_n) = a + b - 5 + 4 + d(u_{n-1}) + d(v_{m-1}) + d(w_{k-1}).$$

We next show that $d(v_2) = d(v_{m-1})$. If $m = 2$, then $v_2 = u_n$, $v_{m-1} = u_1$. So $d(v_2) = d(u_n) = d(u_1) = d(v_{m-1})$. If $m = 3$, clearly, $d(v_2) = d(v_{m-1})$. If $m \geq 4$, then $d(v_2) = d(v_{m-1})$ by Lemma 3.1 (i). Therefore $d(v_2) = d(v_{m-1})$ for all $m \geq 2$.

Similarly, we have $d(w_2) = d(w_{k-1})$ for all $k \geq 2$. Hence $S(u_1) \neq S(u_n)$. On the other hand, $S(u_1) = S(u_n)$ by (2.2), a contradiction.

Subcase 2. $\max\{n, m, k\} \leq 3$. Without loss of generality, suppose that $n = m = 3$ and $k = 2$ or 3 . We claim that $d(u_2) = a + b$. Otherwise, let $d(u_2) = d_{G_0}(u_2) = 2$. Then $d(u_1) + d(u_3) = S(u_2) = S(x_1) = 2 + d(u_3)$ by (2.1) and (2.2), which is impossible since $d(u_1) \geq d_{G_0}(u_1) = 3$. Hence $d(u_2) = a + b$. Similarly, we have $d(v_2) = a + b$.

Note that $d(u_3) \in \{a + b, 5\}$ by Lemma 2.5 (ii). We consider the following two cases:

If $d(u_3) = a + b \geq d_{G_0}(u_3) = 5$. Applying Lemma 2.5 (iv) with $C = u_1 u_2 u_3 v_2 u_1$, we have $d(u_1) \neq a + b$. So $d(u_1) = 3$. Applying (2.3) with $(v, u) = (u_1, x)$ and $(v, u) = (x_1, x)$, respectively. We have $a = \frac{a+b+d(w_2)}{2}$ and $a = 2$, respectively. Hence $d(w_2) = 4 - (a + b) < 0$, a contradiction.

If $d(u_3) = d_{G_0}(u_3) = 5 \neq a + b$, then $a = 7 - (a + b)$ by applying (2.3) with $(v, u) = (x_1, x)$. It follows from Lemma 2.5 (iii) that $a + b = 3$ or 4 . If $a + b = 3$, then $a = 4, b = -1$. Hence $S(u_2) = d(u_1) + 6 = 11$ by (2.1) and (2.2). This is impossible since $d(u_1) = 3$ by Lemma 2.5 (ii). If $a + b = 4$, then $a = 3, b = 1$. Thus $S(u_2) = d(u_1) + 7 = 13$ by (2.1) and (2.2). This is also impossible since $d(u_1) = 3$ or 4 .

Case 2. $l \geq 2$. We show that in this case there is no such graph with exactly two main eigenvalues. By Lemma 3.1 (iii), we have $a = 2$ and $d(x_1) = d(x_2) = 2$. Applying (2.3) with $(v, u) = (x_1, x)$, we have $2 = 2 + d(r_1) - (a + b)$. So $d(r_1) = a + b$. Applying (2.3) with $(v, u) = (r_1, x)$, we have $2 = \frac{a+b-3+4+d(r_2)-(a+b)}{a+b-1}$. Hence $d(r_2) = 2(a + b) - 3$. By Lemma 2.5 (ii), $d(r_2) \in \{2, 4, a + b\}$.

If $d(r_2) = 2$ or 4 , then $a + b$ is not an integer, a contradiction.

If $d(r_2) = a + b$, then $a + b = 3$. So $a = 2, b = 1$. By Lemmas 2.5 (ii) and 3.1 (i), we have $d(r_l) = 4$ and $d(r_{l-1}) = d(r_2) = 3$. Hence $S(r_l) = 3 + d(u_{n-1}) + d(w_{k-1}) + d(v_{m-1}) = 9$ by (2.1) and (2.2). It follows that $d(u_{n-1}) = d(w_{k-1}) = d(v_{m-1}) = 2$. Thus $S(u_{n-1}) = 5$ by (2.2). On the other hand, $S(u_{n-1}) = 4 + d(u_{n-2}) \geq 6$ by (2.1), a contradiction. \square

Lemma 3.5 *Let $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_4$ (see Fig. 1). Then $G = H_i$ for $i = 8, 9, 10$ (see Fig. 3) or $G \in \mathcal{G}_3$ (see Fig. 4).*

Proof. If $G_0 \in \mathcal{T}_4$, then $d_{G_0}(w_1) \in \{3, 4\}$ and so the cycle of G_0 has the length of 3 by Lemmas 2.4 and 3.1 (iii). Hence $G_0 = T_4$ (see Fig. 2), where $k \geq 1$ and $p, q, n, m \geq 2$. For convenience, we set $u_1 = v_1 = s_1, u_n = v_m = t_1$ and $s_p = t_q = w_k$.

If G has no pendent vertex. Without loss of generality, we may assume that $n \geq m, p \geq q$ and consider the following two cases:

Case 1. $k = 1$. Then $S(x_1) = 6$ by (2.1). We claim that $p = 2$. Otherwise, let $p \geq 3$. Then $d(s_2) = 2$ and $S(s_2) = 5$ or 7 by (2.1). So $S(x_1) \neq S(s_2)$. On the other hand, $S(x_1) = S(s_2)$ by (2.2), a contradiction. Hence $p = 2$. Similarly, we have $q = 2, n = 3, m = 2$ or 3 . If $m = 2$, then applying (2.3) with $(v, u) = (w_1, u_2)$ and $(v, u) = (w_1, u_1)$, respectively,

we have $a = 2$ and $a = 1$, respectively. A contradiction. If $m = 3$, then $G = H_8$ (see Fig. 3). By (2.3), H_8 is 2-walk $(2, 2)$ -linear.

Case 2. $k > 1$. By Lemma 2.4, we have $k, p, q, m, n = 2$ or 4.

If $k = 2$, then $S(w_1) = 7 = S(w_2) = 3 + d(s_{p-1}) + d(t_{q-1})$ by (2.1) and (2.2). So $d(s_2) = d(t_2) = 2$. It implies that $p = q = 4$. Hence $n = 4, m = 2$. Otherwise, let $m = n = 4$. Then $S(u_1) = 6 \neq S(w_1) = 7$ by (2.1). On the other hand, $S(u_1) = S(w_1)$ by (2.2), a contradiction. Therefore $n = 4, m = 2$ and $G = H_9$ (see Fig. 3). By (2.3), H_9 is 2-walk $(2, 1)$ -linear.

If $k = 4$, with a similar argument, we have $G = H_{10}$ is 2-walk $(1, 3)$ -linear (see Fig. 3).

If G has at least one pendent vertex, then $k > 1$. Otherwise, suppose that $k = 1$. Then $d(w_1) = 4$ or $a + b$ by Lemma 2.5 (ii).

If $d(w_1) = 4 \neq a + b$. Applying (2.3) with $(v, u) = (x_1, x)$, we have $a = 6 - (a + b)$. It follows from Lemma 2.5 (iii) that $a = 3, b = 0$. So $S(w_1) = 4 + d(s_{p-1}) + d(t_{q-1}) = 12$ by (2.1) and (2.2). This is impossible since $d(s_{p-1}), d(t_{q-1}) = 2$ or 3 by Lemma 2.5 (ii).

If $d(w_1) = a + b \geq 4$. Applying (2.3) with $(v, u) = (x_1, x)$ and $(v, u) = (w_1, x)$, respectively, we have $a = 2$ and $a = \frac{a+b-4+4+d(t_{q-1})+d(s_{p-1})-(a+b)}{a+b-1}$, respectively. Thus

$$d(t_{q-1}) + d(s_{p-1}) = 2(a + b) - 2. \quad (3.3)$$

Note that $d(t_{q-1}), d(s_{p-1}) \in \{2, 3, a + b\}$. We consider the following five cases by symmetry:

If $d(t_{q-1}) = d(s_{p-1}) = 2$, then $a + b = 3$. This contradicts the fact that $a + b \geq 4$.

If $d(t_{q-1}) = 2, d(s_{p-1}) = 3$, then $a + b = 3.5$. This contradicts Lemma 2.2.

If $d(t_{q-1}) = 2, d(s_{p-1}) = a + b$, then $a + b = 4$ by (3.3). Note that $a = 2$. We have $b = 2$ and $d(s_{p-1}) = d(w_1) = 4$. By Lemma 3.1, $d(s_i) = 4$ for $2 \leq i \leq p - 1$. So $S(s_2) = 6 + d(u_1) = 10$ by (2.1) and (2.2). Thus $d(u_1) = 4$. Similarly, we have $d(u_n) = 4$. By (2.1) and (2.2), $S(u_1) = 5 + d(u_2) + d(v_2) = 10$. This is impossible since $d(u_2), d(v_2) = 2$ or 4.

If $d(t_{q-1}) = 3, d(s_{p-1}) = a + b$, then $a + b = 5$ by (3.3). Recall that $a = 2$, we have $b = 3$ and $d(w_1) = 5$. Note that $d(t_{q-1}) = 3 \neq a + b$. We have $q = 2$. Thus $S(t_{q-1}) = 9 = 5 + d(u_{n-1}) + d(v_{n-1})$ by (2.1) and (2.2). By Lemma 2.5 (ii), $d(u_{n-1}), d(v_{n-1}) \in \{2, 3, 5\}$. Hence $d(u_{n-1}) = d(v_{n-1}) = 2$. Therefore $S(u_{n-1}) = 3 + d(u_{n-2}) = 7$, which is impossible since $d(u_{n-2}) \in \{2, 3, 5\}$.

If $d(t_{q-1}) = d(s_{p-1}) = a + b$, then $2(a + b) = 2(a + b) - 2$ by (3.3), a contradiction.

Hence $k > 1$. It follows from Lemma 3.1 (iii) that $a = 2$ and $d(x_1) = d(x_2) = 2$.

Applying (2.3) with $(v, u) = (x_1, x)$, we have $2 = 2 + d(w_1) - (a + b)$. So $d(w_1) = a + b$. Applying (2.3) with $(v, u) = (w_1, x)$, we have $2 = \frac{a+b-3+4+d(w_2)-(a+b)}{a+b-1}$. Thus $d(w_2) = 2(a + b) - 3 \geq 3$. Note that $d(w_2) \in \{2, 3, a + b\}$ by Lemma 2.5 (ii). We have $d(w_2) = 3$ or $a + b$. It follows that $a + b = 3$ and $a = 2, b = 1$.

By Lemma 3.1 (i), $d(w_i) = d(w_2) = 3$ for $2 \leq i \leq k - 1$.

By Lemma 2.5 (ii), $d(w_1) = d(w_k) = d(u_1) = d(u_n) = 3$.

For the vertex w_k , we have $S(w_k) = 3 + d(s_{p-1}) + d(t_{q-1}) = 7$. It follows from Lemma 2.5 (ii) that $d(s_{p-1}) = d(t_{q-1}) = 2$ and $p, q \geq 3$. Hence $p = q = 4$ and $d(s_2) = d(t_2) = 2$ by Lemmas 2.4 and 3.1 (ii).

For the vertex u_1 , we have $S(u_1) = 2 + d(v_2) + d(u_2) = 7$. Note that $d(u_2), d(v_2) \in \{2, 3\}$. We may assume that $d(v_2) = 2, d(u_2) = 3$ by symmetry. It follows from Lemmas 2.4 and 3.1 (ii) that $m = 4, d(v_3) = 2$ and $d(u_i) = 3$ for $2 \leq i \leq n - 1$.

Therefore $G \in \mathcal{G}_3$ (see Fig. 4), where $\max\{k_1, k_2\} \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_3$ is 2-walk $(2, 1)$ -linear.

Up to now, we complete the proof of the Lemma. \square

Lemma 3.6 *There is no graph $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_5$ (see Fig. 1).*

Proof. By way of contradiction, suppose that $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_5$. Let $G_0 = T_5$ (see Fig. 2), where $n, m, k, p, q \geq 2$. For convenience, we set $u_1 = v_1 = s_1, u_n = w_k = t_q, v_m = s_p = w_1 = t_1$ and consider the following two cases:

Case 1. There is a vertex $v \in \{v_{m-1}, s_{p-1}, w_2, t_2\}$ such that $d_{G_0}(v) = 2$ and $d(v) = a + b$. Without loss of generality, let $v = v_{m-1}$. Then $d(v_i) = a + b$ for $2 \leq i \leq m - 1$ by Lemma 3.1 (i). In particular, $a \geq 2, a + b \geq 3$ by Lemma 2.5 (iii). We first show that $d(u_1) = d(w_1) = a + b$.

If $m \geq 4$, then $d(u_1) = d(w_1)$ by Lemma 3.1 (i). Note that $d_{G_0}(u_1) \neq d_{G_0}(w_1)$. We have $d(u_1) = d(w_1) = a + b$ by Lemma 2.5 (ii).

If $m = 3$. Applying (2.3) with $(v, u) = (v_2, x)$, we have

$$a = \frac{a + b - 2 + d(u_1) + d(w_1) - (a + b)}{a + b - 1} = \frac{d(u_1) + d(w_1) - 2}{a + b - 1}.$$

By Lemma 2.5 (ii), $d(u_1) = 3$ or $a + b$ and $d(w_1) = 4$ or $a + b$.

If $d(u_1) = 3, d(w_1) = 4$, then $a = \frac{5}{a+b-1}$. Note that $a \geq 2$ is an integer. We have $a + b = 2$. It contradicts the fact that $a + b \geq 3$.

If $d(u_1) = 3, d(w_1) = a + b \geq 4$, then $a = 1 + \frac{2}{a+b-1} < 2$, a contradiction.

If $d(u_1) = a + b \geq 3, d(w_1) = 4 \neq a + b$, then $a = 1 + \frac{3}{a+b-1}$ is not an integer, a contradiction.

Hence $d(u_1) = d(w_1) = a + b$.

Applying Lemma 2.5 (iv) with $C = u_1 v_2 \dots v_{m-1} w_1 s_{p-1} \dots s_2 u_1$, we have $p \geq 3$ and $d(s_i) \neq a + b$ for some $2 \leq i \leq p - 1$. It follows from Lemmas 2.5 (ii) and 3.1 (i) that $d(s_i) = 2$ for $2 \leq i \leq p - 1$. Applying (2.3) with $(v, u) = (v_2, x)$ and $(v, u) = (u_1, x)$, respectively. We have $a = 2$ and $a = 1 + \frac{d(u_2)}{a+b-1}$, respectively. This together with Lemma 2.5 (ii) and the fact that $a + b \geq d_{G_0}(w_1) = 4$ implies that $d(u_2) = a + b - 1 = d_{G_0}(u_2) \geq 3$. Hence $n = 2, d(u_2) = d_{G_0}(u_2) = 3$. Therefore $a + b = 4$ and $a = b = 2$. By (2.1) and (2.2), $S(u_2) = 4 + d(w_{k-1}) + d(t_{q-1}) = 8$. By Lemma 2.5 (ii), $d(w_{k-1}), d(t_{q-1}) = 2$ or 4 . Thus $d(w_{k-1}) = d(t_{q-1}) = 2$. Hence $S(w_{k-1}) = 3 + d(w_{k-2}) = 6$ by (2.1) and (2.2), which is impossible since $d(w_{k-2}) = 2$ or 4 .

Case 2. For any vertex $v \in \{v_{m-1}, s_{p-1}, w_2, t_2\}$, we have $d_{G_0}(v) = 3$ or $d(v) = 2$. It follows from Lemma 2.4 that $m, k, p, q \leq 4$. Without loss of generality, suppose that $m \geq p, k \geq q$. Hence $m, k = 3$ or 4 .

Subcase 1. $\max\{m, k\} = 4$. Without loss of generality, suppose that $m = 4$. Then $d(u_1) = d(w_1)$ by Lemma 3.1 (i). Note that $d_{G_0}(u_1) \neq d_{G_0}(w_1)$. We have $d(u_1) = d(w_1) = a + b \geq d_{G_0}(w_1) = 4$ by Lemma 2.5 (ii). Thus G has at least one pendent vertex. Applying (2.3) with $(v, u) = (v_2, x)$ and $(v, u) = (u_1, x)$, respectively. We have $a = 2$ and $a = \frac{a+b-3+2+d(s_2)+d(u_2)-(a+b)}{a+b-1}$, respectively. Hence $d(s_2) + d(u_2) = 2(a + b) - 1$.

If $p > 2$, then $d(s_2) = 2$. It follows from the fact that $d_{G_0}(u_2) = 2$ or 3 and $a + b \geq 4$ that $d(u_2) = 2(a + b) - 3 > \max\{a + b, d_{G_0}(u_2)\}$, which is impossible by Lemma 2.5 (ii).

If $p = 2$, then $d(s_2) = d(w_1) = a + b$. So $d(u_2) = a + b - 1 \geq 3$. It follows that $n = 2, d(u_2) = d_{G_0}(u_2) = 3$ and $a + b = 4$. Hence $a = b = 2$. By (2.2), $S(w_{k-1}) = 6$. On the other hand, $S(w_{k-1}) = d(w_{k-2}) + d(u_2) = 5$ or 7 by (2.1), a contradiction.

Subcase 2. $m = k = 3$. Then $d(w_1) + d(u_1) = S(v_2) = S(w_2) = d(w_1) + d(u_n)$ by (2.1) and (2.2). So $d(u_1) = d(u_n)$. Thus $S(u_1) = S(u_n)$ by (2.2).

We claim that $p = q = 2$ or 3 . Otherwise, let $p \neq q$. Without loss of generality, suppose that $p = 3, q = 2$. Then $S(u_1) = d(u_1) - 3 + 4 + d(u_2)$ and $S(u_n) = d(u_n) - 3 + 2 + d(w_1) + d(u_{n-1})$. Note that $d(u_2) = d(u_{n-1}), d(w_1) > 2$. We have $S(u_1) \neq S(u_n)$, a contradiction. Hence $p = q = 2$ or 3 . We consider the following two cases:

Subcase 2.1. G has no pendent vertex. Then $n = 2$. Otherwise, suppose that $n > 2$. Then $d(u_2) = d(v_2) = 2$. By (2.1) and (2.2), $3 + d(u_3) = S(u_2) = S(v_2) = 7$. This is impossible since $d(u_3) = 2$ or 3 . Hence $n = 2$. If $p = q = 2$. Applying (2.3) with $(v, u) = (u_1, v_2)$ and $(v, u) = (w_1, u_1)$, respectively. We have $a = 2$ and $a = 1$, respectively. A contradiction. If $p = q = 3$, also applying (2.3) with $(v, u) = (u_1, v_2)$ and $(v, u) = (w_1, u_1)$, respectively, we have $a = 0$ and $a = 1$, respectively. Also a contradiction.

Subcase 2.2. G has at least one pendent vertex. Then $a \geq 2, a + b \geq 3$.

First, let $p = q = 3$. Applying (2.3) with $(v, u) = (w_1, x)$, we have $a = \frac{d(w_1) - 4 + 8 - (a + b)}{d(w_1) - 1}$. By Lemma 2.5 (ii), $d(w_1) = a + b$ or 4 . It follows that $a < 2$, a contradiction.

Next, suppose that $p = q = 2$. By Lemma 2.5 (ii), $d(w_1) = a + b$ or 4 .

If $d(w_1) = a + b \geq 4$. Applying (2.3) with $(v, u) = (w_1, x)$ and note that $d(u_1) = d(u_n)$, we have $a = \frac{2d(w_1)}{a + b - 1}$. If $d(u_1) = a + b$, then $a = 2 + \frac{2}{a + b - 1}$ is not an integer, which is impossible by Lemma 2.2. If $d(u_1) = d_{G_0}(u_1) = 3 \neq a + b$, then $a + b = 4$ and $a = b = 2$. So $S(v_2) = 6$ by (2.2). On the other hand, $S(v_2) = d(u_1) + d(w_1) = 7$ by (2.1), a contradiction.

If $d(w_1) = 4 \neq a + b$. Applying (2.3) with $(v, u) = (v_2, x)$, we have $a = 4 + d(u_1) - (a + b)$. If $d(u_1) = a + b$, then $a = 4$. For the vertex u_1 , we have $S(u_1) = 4(a + b) + b = a + b - 3 + 6 + d(u_2)$. It follows that $d(u_2) = 4(a + b) - 7 > \max\{a + b, d_{G_0}(u_2)\}$, which is impossible by Lemma 2.5 (ii). If $d(u_1) = 3 \neq a + b$, then $a = 7 - (a + b)$. Note that $a + b \neq 3, 4$. We have $a + b = 5$ and $a = 2, b = 3$ by Lemma 2.5 (iii). By (2.1) and (2.2), $S(w_1) = 11 = 4 + d(u_1) + d(u_n)$. This is impossible since $d(u_1) = d(u_n)$.

Up to now, we have completed the proof of the Lemma. \square .

Lemma 3.7 *Let $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_6$. Then $G = H_i$ for $i = 11, \dots, 15$ (see Fig. 3) or $G \in \mathcal{G}_4$ (see Fig. 4).*

Proof. Let $G_0 = T_6$ (see Fig. 2), where $n, m, p, k \geq 2$. For convenience, we set $u_1 = v_1 = w_1 = s_1$ and $u_n = v_m = w_k = s_p$.

Case 1. There is a vertex $v \in \{u_2, v_2, w_2, s_2\}$ such that $d_{G_0}(v) = 2$ and $d(v) = a + b$. Without loss of generality, suppose that $v = u_2$. Applying (2.3) with $(v, u) = (u_2, x)$, we have

$$a = \frac{a + b - 2 + d(u_1) + d(u_3) - (a + b)}{a + b - 1} = \frac{d(u_1) + d(u_3) - 2}{a + b - 1}. \quad (3.4)$$

By Lemmas 2.5 (ii), $d(u_1), d(u_3) = 4$ or $a + b$. We consider the following three cases:

Subcase 1. $d(u_1) = d(u_3) = a + b$, then $a = 2$ by (3.4).

We now show that $d(u_i) = a + b$ for $1 \leq i \leq n$. If $n = 3$, then obviously, $d(u_i) = a + b$ for $1 \leq i \leq n$. If $n \geq 4$, then $d(u_i) = d(u_2) = a + b$ for $1 < i < n$ and $d(u_n) = d(u_1) = a + b$ by Lemma 3.1 (i). Hence $d(u_i) = a + b$ for $1 \leq i \leq n$.

Applying Lemma 2.5 (iv) with $C = u_1 u_2 \dots u_n v_{m-1} \dots v_2 u_1$, we have $d(v_i) \neq a + b$ for some $2 \leq i \leq m - 1$. It follows from Lemmas 2.5 (ii) and 3.1 (i) that $m \geq 3$ and $d(v_i) = 2$ for all $2 \leq i \leq m - 1$. Thus $S(v_2) = 2a + b = a + b + d(v_3)$. Note that $a = 2$. We have $d(v_3) = 2$. So $m \geq 4$. It follows from Lemma 3.1 (ii) that $m = 4$. Similarly, we have $k = p = 4$ and $d(w_2) = d(w_3) = d(s_2) = d(s_3) = 2$.

For the vertex u_1 , $S(u_1) = 2(a + b) + b = a + b - 4 + 6 + a + b$ by (2.1) and (2.2). Hence $b = 2$. It follows that $d(u_i) = a + b = 4$ for $1 \leq i \leq n$.

Therefore $G \in \mathcal{G}_4$ (see Fig. 4), where $l_1 \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_4$ is 2-walk (2, 2)-linear.

Subcase 2. $d(u_1) = a + b, d(u_3) = 4$ or $d(u_1) = 4, d(u_3) = a + b$ and $a + b \neq 4$. Then $a = 1 + \frac{3}{a + b - 1}$ is not an integer by (3.4), a contradiction.

Subcase 3. $d(u_1) = d(u_3) = 4 \neq a + b$. Then $n = 3$ and $a = \frac{6}{a+b-1}$. It follows from Lemma 2.5 (iii) that $a + b = 3$ and $a = 3, b = 0$. Thus $d(u_2) = a + b = 3$. For the vertex u_1 , $S(u_1) = 12 = 3 + d(v_2) + d(w_2) + d(s_2)$ by (2.1) and (2.2). Note that $d(v_2), d(w_2), d(s_2) \in \{2, 4, a + b\}$. We have $d(v_2) = d(w_2) = d(s_2) = a + b = 3$. So $S(v_2) = 9 = 5 + d(v_3)$. Thus $d(v_3) = 4 \neq a + b$. It implies that $m = 3$. Similarly, we have $k = p = 3$. Therefore $G = H_{11}$ (see Fig. 3). By (2.3), H_{11} is 2-walk (3, 0)-linear.

Case 2. For any vertex $v \in \{u_2, v_2, w_2, s_2\}$, $d_{G_0}(v) = 4$ or $d(v) = 2$. By Lemma 2.4, $n, m, p, k \leq 4$. Without loss of generality, suppose that $n \geq m \geq p \geq k$. Then $n \geq m \geq p \geq 3$ and $d(u_2) = d(v_2) = d(s_2) = 2$. By (2.1) and (2.2), $S(u_2) = d(u_1) + d(u_3) = S(v_2) = d(u_1) + d(v_3)$. Thus $d(v_3) = d(u_3)$. It implies that $m = n$. Similarly, we have $p = n$ and $k = n$ or 2. Hence $k = p = m = n \in \{3, 4\}$ or $k = 2, p = m = n \in \{3, 4\}$.

If G has no pendent vertex, then $G = H_i$ for $i = 12, \dots, 15$ (see Fig. 3). By (2.3), H_{12} is 2-walk (1, 6)-linear, H_{13} is 2-walk (0, 8)-linear, H_{14} is 2-walk (2, 2)-linear and H_{15} is 2-walk (1, 4)-linear.

If G has at least one pendent vertex x , then $x \in N(u_1)$ or $x \in N(u_n)$. Without loss of generality, suppose that $x \in N(u_1)$. Then $d(u_1) = a + b \geq 5$. Applying (2.3) with $(v, u) = (u_1, x)$, we have $a = \frac{d(w_2)+2}{a+b-1}$. By Lemma 2.5 (ii), $d(w_2) = 2, 4$ or $a + b$. It implies that $a < 2$. This is impossible by Lemma 2.5 (iii).

Up to now, we have completed the proof of the Lemma. \square

Lemma 3.8 *Let $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_7$ (see Fig. 1). Then $G = H_i$ for $i = 16, \dots, 23$ (see Fig. 3) or $G \in \mathcal{G}_j$ for $j = 5, 6$ (see Fig. 4).*

Proof. Let $G_0 = T_7$ (see Fig. 2), where $n, m, k, l, p, q \geq 2$. For convenience, we set $u_1 = w_1 = s_1, u_n = w_k = t_1, v_1 = r_1 = s_p$ and $v_m = r_l = t_q$.

If G has no pendent vertex, then $n, m, k, l, p, q \leq 4$ by Lemma 2.4. Without loss of generality, suppose that $n \geq k, m \geq l, p \geq q$. Then $n = 3$ or 4. By (2.1) and (2.2), $S(u_1) = d(u_2) + d(w_2) + d(s_2) = S(u_n) = d(u_{n-1}) + d(w_{k-1}) + d(t_2)$. Note that $d(u_2) = d(u_{n-1}), d(w_2) = d(w_{k-1})$. We have $d(s_2) = d(t_2)$. It implies that $p = q$. Similarly, we have $k = l$. If $n = 3$, then $m = 3, k, p = 2$ or 3 by Lemma 2.4. Hence $G = H_i$ for $i = 16, 17, 18, 19$ (see Fig. 3). If $n = 4$, then $m = 4, k, p = 2$ or 4 by Lemma 2.4. Hence $G = H_j$ for $j = 20, 21, 22, 23$ (see Fig. 3). By (2.3), H_{16} is 2-walk (2, 2)-linear, H_{17} and H_{18} are 2-walk (1, 4)-linear, H_{19} is 2-walk (0, 6)-linear, H_{20} is 2-walk (3, -1)-linear, H_{21} and H_{22} are 2-walk (2, 1)-linear, H_{23} is 2-walk (1, 3)-linear.

If G has at least one pendent vertex. Then $a \geq 2, a + b \geq 3$.

We first show that $d(v_1) = d(v_m)$.

If $\max\{m, l\} \geq 4$, then $d(v_1) = d(v_m)$ by Lemma 3.1 (i).

If $m, l \leq 3$. By way of contradiction, suppose that $d(v_1) \neq d(v_m)$. By Lemma 2.5 (ii), we may assume that $d(v_1) = a + b > 3, d(v_m) = 3$ without loss of generality. Applying (2.3) with $(v, u) = (v_1, v_3)$, we have

$$a = \begin{cases} \frac{d(s_{p-1}) - d(t_{q-1})}{a+b-3} & \text{if } m = 3, l = 2 \text{ or } m = 2, l = 3, \\ 1 + \frac{d(s_{p-1}) - d(t_{q-1})}{a+b-3} & \text{if } m = 3, l = 3. \end{cases}$$

By Lemma 2.5 (ii), $d(s_{p-1}), d(t_{q-1}) = 2, 3$ or $a + b$.

If $m = 3, l = 2$ or $m = 2, l = 3$, then $d(s_{p-1}) = a + b, d(t_{q-1}) = 2$ since $a \geq 2, a + b \geq 3$. So $a = 1 + \frac{1}{a+b-3}$. It implies that $a + b = 4$ and $a = b = 2$. By (2.1) and (2.2), $S(v_3) = 6 + d(v_2) = 8$. So $d(v_2) = 2$ and $S(v_2) = 6$ by (2.2). On the other hand, $S(v_2) = 7$ by (2.1), a contradiction.

If $m = l = 3$, then $d(s_{p-1}) = a + b \neq 3, d(t_{q-1}) = 3$ or $d(s_{p-1}) = a + b, d(t_{q-1}) = 2$ since $a \geq 2, a + b \geq 3$. If $d(s_{p-1}) = a + b \neq 3, d(t_{q-1}) = 3$, then $a + b \geq 4$ and $a = 2$. Thus $b \geq 2$. By (2.1) and (2.2), $S(v_3) = 6 + b = d(v_2) + d(r_2) + 3$, which is impossible since $d(v_2), d(r_2) = 2$ or $2 + b$ and $b \geq 2$. If $d(s_{p-1}) = a + b, d(t_{q-1}) = 2$, then $a = 2 + \frac{1}{a+b-3}$. It implies that $a + b = 4$ and $a = 3, b = 1$. By (2.1) and (2.2), $S(v_3) = d(v_2) + d(r_2) + 2 = 10$. Note that $d(v_2), d(r_2) = 2$ or 4 by Lemma 2.5 (ii), we have $d(v_2) = d(r_2) = 4$. Thus $S(v_2) = 13$ by (2.2). On the other hand, $S(v_2) = 9$ by (2.1), a contradiction.

Hence $d(v_1) = d(v_m)$.

Next, we show that $d(v_1) = d(v_m) = a + b$. On the contrary, suppose that $d(v_1) = d(v_m) = 3 \neq a + b$ by Lemma 2.5 (ii). Then $a + b \geq 4$. Note that $\max\{m, l\} \geq 3$. Without loss of generality, suppose that $m \geq 3$. By Lemma 2.5 (ii), we have $d(v_2) = a + b$ or 2 .

If $d(v_2) = a + b$. Applying (2.3) with $(v, u) = (v_2, x)$, we have $a = \frac{d(v_3)+1}{a+b-1}$. By Lemma 2.5 (ii), $d(v_3) = 2, 3$ or $a + b$. This together with $a + b \geq 4$ implies that $a < 2$. It contradicts the fact that $a \geq 2$.

If $d(v_2) = 2$. Applying (2.3) with $(v, u) = (v_2, x)$, we have $a = 3 + d(v_3) - (a + b)$. If $m \geq 4$, then $d(v_3) = 2$ by Lemma 3.1 (i). Note that $a + b \geq 4$. We have $a = 5 - (a + b) \leq 1$. It contradicts the fact $a \geq 2$. Thus $m = 3$. Hence $d(v_3) = 3$ and $a = 6 - (a + b)$. Note that $a + b \geq 4, a \geq 2$. We have $a + b = 4$ and $a = b = 2$. By (2.1) and (2.2), $S(v_1) = 2 + d(r_2) + d(s_{p-1}) = 8$. So $d(r_2) + d(s_{p-1}) = 6$.

If $l = 2$, then $d(r_2) = d(v_m) = 3$. So $d(s_{p-1}) = 3 \neq a + b$. It implies that $p = 2$ and $d(u_1) = d(s_{p-1}) = 3$. Similarly, we get $q = 2$ and $d(u_n) = 3$. By (2.1) and (2.2), $S(u_1) = 3 + d(w_2) + d(u_2) = 8$. Note that $d(w_2), d(u_2) = 2, 3$ or 4 by Lemma 2.5 (ii). It follows that $d(u_2) = 2, d(w_2) = 3$ or $d(u_2) = 3, d(w_2) = 2$. We may suppose that $d(u_2) = 2, d(w_2) = 3$ without loss of generality. Then $k = 3$ and $d(u_i) = 2$ for $2 \leq i \leq n - 1$ by Lemma 3.1 (i). Hence G has no pendent vertex. This is a contradiction.

If $l \geq 3$, then $d(r_2) = 2$ or 4 . If $d(r_2) = 2$, then $d(s_{p-1}) = 6 - d(r_2) = 4$. Thus $S(s_{p-1}) = 10$ by (2.2). This is impossible since $S(s_{p-1}) = 4 - 2 + 3 + d(s_{p-2}) \leq 9$ by (2.1). If $d(r_2) = 4$, then $S(r_2) = 10$ by (2.2). This is also impossible since $S(r_2) = 4 - 2 + 3 + d(r_3) \leq 9$ by (2.1).

Hence $d(v_1) = d(v_m) = a + b$. Dually, we have $d(u_1) = d(u_n) = a + b$.

Applying Lemma 2.5 (iv) with $C = v_1 v_2 \dots v_m r_{l-1} \dots r_2 v_1$, we have, without loss of generality, $m \geq 3$ and $d(v_i) \neq a + b$ for some $2 \leq i \leq m - 1$. So $d(v_i) = 2$ for all $2 \leq i \leq m - 1$ by Lemmas 2.5 (ii) and 3.1.

Applying (2.3) with $(v, u) = (v_1, x)$ and $(v, u) = (v_2, x)$, respectively. We have

$$a = \frac{d(r_2) + d(s_{p-1}) - 1}{a + b - 1} \text{ and } a = d(v_3), \quad (3.5)$$

respectively.

We claim that $m \geq 4$. Otherwise, suppose that $m = 3$. Then $a = d(v_3) = a + b$ and $d(r_2) + d(s_{p-1}) = a(a + b - 1) + 1 \geq 3(a + b) - 2 > 2(a + b)$, which is impossible since $d(r_2), d(s_{p-1}) = 2$ or $a + b$ by Lemma 2.5 (ii). Hence $m \geq 4$. Therefore $m = 4$ by Lemma 3.1 (ii). Thus $a = d(v_3) = 2$. It follows from (3.5) that $d(r_2) + d(s_{p-1}) = 2(a + b) - 1$. By Lemma 2.5 (iii), $d(r_2), d(s_{p-1}) = 2$ or $a + b$.

If $d(r_2) = d(s_{p-1}) = 2$, then $a + b = 2.5$, a contradiction.

If $d(r_2) = d(s_{p-1}) = a + b$, then $2(a + b) = 2(a + b) - 1$, a contradiction.

If $d(r_2) = 2, d(s_{p-1}) = a + b$, then $a + b = 3$. So $a = 2, b = 1$.

By Lemma 3.1 (i), $d(s_i) = d(s_{p-1}) = 3$ for $2 \leq i \leq p - 1$.

For the vertex u_1 , $S(u_1) = 3 + d(u_2) + d(w_2) = 7$ by (2.1) and (2.2). By Lemma 2.5 (ii), $d(u_2), d(w_2) = 2$ or 3 . It follows that $d(u_2) = d(w_2) = 2$. So $S(u_2) = 3 + d(u_3) = 5$. Thus

$d(u_3) = 2$ and $n \geq 4$. Hence $n = 4$ by Lemma 3.1 (ii).

Similarly, we have $d(w_3) = d(r_2) = 2$ and $k = l = 4$.

By (2.1) and (2.2), $S(u_4) = 4 + d(t_2) = 7$. So $d(t_2) = 3$. Hence $d(t_i) = 3$ for $1 \leq i \leq q-1$ by Lemma 3.1 (i).

Therefore $G \in \mathcal{G}_5$ (see Fig. 4), where $\max\{k_1, k_2\} \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_5$ is 2-walk $(2, 1)$ -linear.

If $d(r_2) = a+b, d(s_{p-1}) = 2$, then with a similar argument of the case $d(r_2) = 2, d(s_{p-1}) = a+b$, we have $G \in \mathcal{G}_6$ is 2-walk $(2, 1)$ -linear (see Fig. 4), where $\max\{k_1, k_2\} \geq 1$.

Up to now, we have complete the proof of the Lemma. \square

Lemma 3.9 *Let $G \in \mathcal{G}$ with $G_0 \in \mathcal{T}_8$. Then $G \cong H_i$ for $i = 24, \dots, 30$ (see Fig. 3) or $G \in \mathcal{G}_j$ for $j = 7, 8$ (see Fig. 4).*

Proof. Let $G_0 = T_8$ (see Fig. 2), where $n, m, k, l, p, q \geq 2$. For convenience, we set $u_1 = v_1 = w_1, u_n = t_1 = s_p, v_m = r_1 = t_q$ and $w_k = s_1 = r_l$.

Case 1. There is a vertex $v \in \{w_2, v_2, u_2\}$ with $d_{G_0}(v) = 2$ and $d(v) = a+b$. Without loss of generality, suppose that $v = w_2$. Then $k \geq 3$ and $d(w_i) = a+b$ for $2 \leq i \leq k-1$ by Lemma 3.1 (i). In particular, $a \geq 2, a+b \geq 3$ by Lemma 2.5 (iii).

Applying Lemma 2.3 with $(v, u) = (w_2, x)$, we have $a = \frac{d(w_1)+d(w_3)-2}{a+b-1}$. By Lemma 2.5 (ii), we have $d(w_1), d(w_3) = 3$ or $a+b$.

If $d(w_1) = d(w_3) = 3 \neq a+b$, then $a+b \geq 4$ and $a = \frac{4}{a+b-1} < 2$, a contradiction.

If $d(w_1) = a+b, d(w_3) = 3$ or $d(w_3) = a+b, d(w_1) = 3$ and $a+b \neq 3$, then $a = \frac{a+b+1}{a+b-1} = 1 + \frac{2}{a+b-1} < 2$, a contradiction.

If $d(w_1) = d(w_3) = a+b$, then $a = 2$. We claim that $a+b = 3$. Otherwise, let $a+b > 3$. For the vertex $w_1, S(w_1) = 2(a+b) + b = a+b-3 + a+b + d(u_2) + d(v_2)$. Thus $d(u_2) + d(v_2) = a+b+1$. By Lemma 2.5 (ii), $d(u_2), d(v_2) = 2, 3$ or $a+b$. Note that $a+b > 3$. We have $d(u_2) = 2, d(v_2) = 3$ or $d(u_2) = 3, d(v_2) = 2$ or $d(u_2) = d(v_2) = 3$. Without loss of generality, we consider the following two cases:

If $d(u_2) = 2, d(v_2) = 3$, then $a+b = 4$ and $m = 2$. Note that $a = 2$. We have $b = 2$. If $k \geq 4$, then $d(w_k) = d(w_1) = 4$ by Lemma 3.1 (i). If $k = 3$, then $d(w_k) = d(w_3) = 4$. So $d(r_3) = 2$ or 4 by Lemma 2.5 (ii). On the other hand, $S(v_2) = 4 + d(r_2) + d(t_{q-1}) = 8$ by (2.1) and (2.2). So $d(r_2) = d(t_{q-1}) = 2$. By (2.1) and (2.2), $S(r_2) = 6 = 3 + d(r_3)$. Thus $d(r_3) = 3$, a contradiction.

If $d(u_2) = d(v_2) = 3$, then $a+b = 5$ and $m = n = 2$. Note that $a = 2$. We have $b = 3$. By (2.1) and (2.2), $S(v_2) = 5 + d(r_2) + d(t_{q-1}) = 9$. Thus $d(r_2) = d(t_{q-1}) = 2$. Hence $S(r_2) = 7 = 3 + d(r_3)$. This is impossible since $d(r_3) \in \{2, 3, 5\}$.

Therefore $a+b = 3$ and $a = 2, b = 1$.

By (2.1) and (2.2), $S(w_1) = 7 = 3 + d(v_2) + d(u_2)$. So $d(v_2) = d(u_2) = 2$. Hence $S(v_2) = 5 = 3 + d(v_3)$. It follows that $d(v_3) = 2$ and $m \geq 4$. Therefore $m = 4$ by Lemma 3.1 (ii).

Similarly, we have $n = l = p = 4, d(u_2) = d(u_3) = d(r_2) = d(r_3) = d(s_2) = d(s_3) = 2$ and $d(t_i) = 3$ for $2 \leq i \leq q-1$.

Therefore $G \in \mathcal{G}_8$ (see Fig. 4), where $\max\{k_1, k_2\} \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_8$ is 2-walk $(2, 1)$ -linear.

Case 2. For any vertex $v \in \{w_2, v_2, u_2\}$, we have $d_{G_0}(v) = 3$ or $d(v) = 2$.

If G has no pendent vertex, then $n, m, k, l, p, q \leq 4$ by Lemma 2.4. If $n = m = k = l = p = q = 2$, then G is regular. It is well known that a graph is regular if and only if it has exactly one main eigenvalues. Thus $\max\{n, m, k, p, q\} \geq 3$. Without loss of generality, suppose that $n \geq 3$. We consider the following two cases:

Subcase 1. $n = 3$. Then $S(u_2) = 6$ and $m, k, l, p, q = 2$ or 3 by Lemma 2.4.

If $m = k = 2$, then $S(u_1) = 8 = S(u_3) = 2 + d(t_2) + d(s_{p-1})$. So $d(t_2) = d(s_{p-1}) = 3$ and hence $p = q = 2$. Similarly, $S(u_1) = 8 = S(v_2) = 6 + d(r_2)$. Thus $d(r_2) = 2$ and $l = 3$. Therefore $G = H_{24}$ (see Fig. 3). By (2.3), H_{24} is 2-walk $(2, 2)$ -linear. With a similar argument, we have:

If $m = 2, k = 3$ or $m = 3, k = 2$, then $G = H_{25}$ is 2-walk $(1, 4)$ -linear (see Fig. 3).

If $m = k = 3$, then $G = H_{26}$ is 2-walk $(0, 6)$ -linear (see Fig. 3).

Subcase 2. $n = 4$. Then $m, k, l, p, q = 2$ or 4 by Lemma 2.4. With a similar argument of Subcase 1, we have the following cases:

If $m = k = 2$, then $G = H_{27}$ is 2-walk $(3, -1)$ -linear (see Fig. 3).

If $m = 2, k = 4$ or $m = 4, k = 2$, then $G = H_{28}$ is 2-walk $(2, 1)$ -linear (see Fig. 3).

If $m = k = 4$, then $G = H_{29}$ is 2-walk $(1, 3)$ -linear (see Fig. 3).

If G has at least one pendent vertex x , then $x \in N(v)$ for $v \in \{w_1, v_m, u_n, w_k, r_{i_1}, t_{i_2}, s_{i_3}\}$, where $2 \leq i_1 \leq l - 1, 2 \leq i_2 \leq q - 1, 2 \leq i_3 \leq p - 1$. If $v \in \{r_{i_1}, t_{i_2}, s_{i_3}\}$, then with a similar argument of Case 1, we have $G \in \mathcal{G}_8$. Hence we may suppose that $v \in \{w_1, v_m, u_n\}$. In particular, assume that $x \in N(w_1)$ without loss of generality. Then $d(w_1) = a + b \geq 4$ and $d(u) = d_{G_0}(u)$ for $u \in \{r_{i_1}, t_{i_2}, s_{i_3}\}$, where $2 \leq i_1 \leq l - 1, 2 \leq i_2 \leq q - 1, 2 \leq i_3 \leq p - 1$. Applying Lemma (2.3) with $(v, u) = (w_1, x)$, we have

$$a = \frac{a + b - 3 + d(u_2) + d(v_2) + d(w_2) - (a + b)}{a + b - 1} = \frac{d(u_2) + d(v_2) + d(w_2) - 3}{a + b - 1}.$$

Note that $d(u_2), d(v_2), d(w_2) = 2, 3$ or $a + b$ and $a + b \geq 4$. We consider the following cases without loss of generality:

If $d(u_2) = d(v_2) = d(w_2) = 2$, then $a = \frac{3}{a+b-1} < 2$, a contradiction.

If $d(u_2) = d(v_2) = 2, d(w_2) = 3$, then $a = \frac{4}{a+b-1} < 2$, a contradiction.

If $d(u_2) = d(v_2) = 2, d(w_2) = a + b$, then $a = 1 + \frac{2}{a+b-1} < 2$, a contradiction.

If $d(u_2) = 2, d(v_2) = d(w_2) = 3$, then $a = \frac{5}{a+b-1} < 2$, a contradiction.

If $d(u_2) = 2, d(v_2) = 3, d(w_2) = a + b$, then $m = 2$ and $a = 1 + \frac{3}{a+b-1}$. By Lemmas 2.2 and 2.5 (ii), we have $a + b = 4$ and $a = 2$. Hence $d(w_i) = d(w_2) = 4$ for $2 \leq i \leq k - 1$. By (2.1) and (2.2), $S(w_{k-1}) = 4 + 2 + d(w_k) = 10$. Thus $d(w_k) = 4$. Similarly, $S(v_2) = 4 + d(r_2) + d(t_{q-1}) = 8$. It implies that $d(r_2) = d(t_{q-1}) = 2$. Thus $S(r_2) = 6 = 3 + d(r_3)$. Hence $d(r_3) = 3$. On the other hand, $d(r_3) \in \{2, 4\}$ by Lemma 2.5 (ii) and the fact that $d(w_k) = 4$. This is a contradiction.

If $d(u_2) = 2, d(v_2) = d(w_2) = a + b$, then $a = 2 + \frac{1}{a+b-1}$ is not an integer, a contradiction.

If $d(u_2) = d(v_2) = d(w_2) = 3$, then $n = m = k = 2$ and $a = \frac{6}{a+b-1}$. It follows from Lemma 2.5 (iii) that $a + b = 4$ and $a = b = 2$. By (2.1) and (2.2), $S(v_2) = 4 + d(r_2) + d(t_{q-1}) = 8$. It implies that $d(r_2) = d(t_{q-1}) = 2$. Hence $S(r_2) = 3 + d(r_3) = 6$. It follows that $d(r_3) = 3$ and $l = 3$. Similarly, we have $p = q = 3$. Therefore $G = H_{30}$ (see Fig. 3). By (2.3), H_{30} is 2-walk $(2, 2)$ -linear.

If $d(u_2) = d(v_2) = 3, d(w_2) = a + b$, then $n = m = 2$ and $a = 1 + \frac{4}{a+b-1}$. Thus $a + b = 5$ and $a = 2$ by Lemma 2.5 (iii). Hence $d(w_1) = d(w_2) = a + b = 5$. For the vertex v_2 , $S(v_2) = 5 + d(r_2) + d(t_{q-1}) = 9$. It implies that $d(r_2) = d(t_{q-1}) = 2$. Thus $S(r_2) = 3 + d(r_3) = 7$, which is impossible since $d(r_3) \in \{2, 3, 5\}$ by Lemma 2.5 (ii).

If $d(u_2) = 3, d(v_2) = d(w_2) = a + b$, then $a = 2 + \frac{2}{a+b-1}$ is not an integer, a contradiction.

If $d(u_2) = d(v_2) = d(w_2) = a + b$, then $n = m = k = 2$ and $a = 3$. We claim that $p = l = 2$. Otherwise, let $p, l > 2$. By (2.1), $S(w_1) = a + b - 3 + 3(a + b), S(w_2) = a + b - 3 + a + b + d(s_2) + d(r_{l-1})$. Note that $d(s_2), d(r_{l-1}) = 2$ by assumption. We have $S(w_1) > S(w_2)$. On the other hand, $S(w_1) = S(w_2)$ by (2.2), a contradiction. Hence

$p = l = 2$. Similarly, we have $q = 2$. Thus $G \in \mathcal{G}_7$ (see Fig. 4), where $b \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_7$ is 2-walk $(3, b)$ -linear.

Up to now, we have complete the proof of the Lemma. \square

Theorem 3.10 *The graphs H_i for $i = 1, \dots, 30$ and those in \mathcal{G}_j for $j = 1, \dots, 8$ are all connected tricyclic graphs with exactly two main eigenvalues.*

Proof. It follows directly from Lemmas 3.2–3.9. \square

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